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On the thermo-elasto-statics of composites with coated randomly distributed inclusions

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Abstract

We consider a linearly elastic composite medium with stress free strains, which consists of a homogeneous matrix containing a homogeneous and statistically uniform random set of coated ellipsoidal inclusions having all the same form, orientation and mechanical properties. We are using the main hypothesis of many micromechanical methods, according to which each inclusion is located inside a homogeneous so-called effective field. It is shown, in the framework of the effective field hypothesis, that from a solution of the pure elastic problem (with zero stress free strains) for the composite the relations for effective thermal expansions, stored energy and average thermoelastic strains inside the components can be found. This way one obtains the generalization of the classical formulae by Rosen and Hashin (1970. Int. J. Eng. Sci. 8, 157–173), which are exact for two-component composites. The proposed theory is applied to the example of composites reinforced with particles with thin inhomogeneous (along inclusion surface) coatings. For a single coated inclusion the micromechanical approach is based on the Green function technique as well as on the interfacial Hill operators. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

More detailed considerations of the mechanical behavior of composite materials require the analysis of the interface between the reinforcement and the matrix. These interfaces may represent: weak interfacial layer due to imperfect bonding between the two phases; inter-diffusion and/or chemical interaction zones (with properties varying through the thickness and/or along the surface) at the interface between the two phases. It is well known that the overall effective properties of composite

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materials are significantly influenced by the properties of the interfaces between the constituents. First, the interface controls the in situ reinforcement's (particles or fibers) strength and hence the strength of the composite. Secondly, defects and damage are likely to occur at the interface (for example debonding, sliding and interface cracks, etc.) and these interfacial defects control the degradation of the composite. Therefore, to evaluate more accurately the effective properties of a composite, the behavior and structures of interfaces must be taken into consideration (Qu, 1993; Cherkaoui et al., 1995). In our short survey at first we consider the problem of a single coated inclusion inside an infinite matrix and then different homogenization schemes are discussed.

Classical works dealing with three-phase solids with spherical or cylindrical coated inclusions are discussed, e.g. by Hashin (1962), Hashin and Rosen (1964). More general cases of mechanical loading, including location dependent transformation were considered by Christensen and Lo (1979), Luo and Weng (1987), (see references in Benveniste et al., 1989). Theocaris (1987) as well as Jasiuk and Kouider (1993) analyzed the effect of the variation of elastic properties with the radial distance from the fiber's boundary in continuously reinforced fiber composites. The above studies are restricted to isotropic materials and spherical or cylindrical reinforcement shapes. Micata and Taya (1986) applied Boussinesq–Sadowsky stress-functions when calculating the stress field for two confocal prolate spheroids embedded in an infinite body.

The thin-layer hypothesis appeared as a principal step in the investigation of coated inclusions, becasue it allows the use of the well-developed Eshelby (1961) theory and Hill (1983) interface operators for the general case of anisotropy of the materials being in contact. In his pioneering paper in this direction Walpole (1978) assumed that the stress and strain components inside the inclusion coincide with those already determined before the coating was introduced. Afterwards this assumption was replaced by the hypothesis of homogeneity of the stress state inside the core (inclusion) and thermoelastic problems were considered (Hatta and Taya, 1987; Qiu and Weng, 1991; Chang and Cheng, 1992; Cherkaoui et al., 1995). In most of the papers homogeneous thermo-elastic properties were assumed for the coating. In the present work we relax these restrictive assumptions. The case of inhomogeneity of elastic and mismatch properties in the coating is a typical situation due to the production of the coated inclusions and due to thermal and plastic deformations of the matrix near the inclusion. Even in situations in which for a specific reference system (connected to the unit normal and tangential vectors of the surface of the inclusion) homogeneous stress and strain fields may be assumed, the introduction of a global coordinate system requires the consideration of inhomogeneous fields in the coating.

Another direction of research in the field of coated inclusion mechanics is dealing with sliding interfaces being intensively treated by Mura (1987), Hashin (1991), Jasiuk et al. (1988), Hashin (1991), Dvorak and Benveniste (1992a), Qu (1992), Huang et al. (1993). The elastic field of a single sliding inclusion is solved by distributing Somiliana's dislocation on the interface where the interface shear exceeds the stick limit and the contact condition changes from perfect bonding to perfect sliding with non-continuous elastic fields. Furthermore, Hashin (1991) has shown that sliding along a two-dimensional interface is equivalent to the response of some isotropical very thin flexible coating. Of course, all these problems and a number of others can be solved by numerical methods which, however, are not discussed in this paper.

The solution of the problem of a single coated inclusion in an infinite matrix is used to treat subsequently the analysis of composites with a random set of such inclusions. A considerable number of methods is known in the linear theory of composites with homogeneous inclusions, which yield the effective thermoelastic constants and stress field averages in the components. A classification of these methods was proposed by Willis (1983). Many references are provided by the reviews of Willis (1982, 1983), Mura (1987), Kreher and Pompe (1989), Buryachenko and Parton (1992c), Nemat-Nasser and Hori (1993). Nowadays, it appears that variants of the effective medium (Kröner, 1961; Hill, 1965) and mean field methods by Mori and Tanaka (1973) (see also Benveniste, 1987) are the most popular and widely used methods. Despite the fact that these methods have certain drawbacks (discussed by Norris, 1989; Benveniste et al., 1991; Qiu and Weng, 1990; Buryachenko and Parton, 1990b), they provide relatively simple analytical estimations. Recently a new method has become known by the open literature, namely the multiparticle effective field method (MEFM) was put forward and developed by Buryachenko (1987), Buryachenko and Lipanov (1986a, b) (more references may be found in the survey of Buryachenko and Parton, 1992c; Buryachenko and Kreher, 1995; Buryachenko, 1996). The MEFM is based on the theory of functions of random variables and Green's functions. Within this method one constructs a hierarchy of statistical moment equations for conditional averages of the stresses in the inclusions. The hierarchy is then cut by introducing the notion of an effective field. This way the interaction of different inclusions is taken into account. Thus, the MEFM does not make use of a number of hypotheses which form the basis of the traditional one-particle methods. Buryachenko and Parton (1990b, c) demonstrated that the MEFM includes as particular cases the well-known methods of mechanics of strongly heterogeneous media (such as the effective medium and the mean field methods).

A large number of papers exist with the applications of the above mentioned methods for the analysis of composites with different shapes, orientations and elastic properties of homogeneous ellipsoidal inclusions. The solution of the problem of coated inclusions and the consideration of different constitutive relations (thermal flow, thermo-elasticity, electro-magneto-elasticity and so on) have been leading to an increase in the number of 'generalized' and 'modified' versions of these classical methods. However, it is a fundamental feature of the following general results, that all these indicated methods (effective medium, mean field method, MEFM and some others) are based on the same so-called 'effective field hypothesis', according to which each inclusion has an ellipsoidal shape and is located in some effective field, which is homogeneous over the considered inclusion. In this paper for the case of coated inclusions with the same shape, orientation and mechanical properties, it is shown that from a solution of a purely elastic problem (with zero stress free strains) the relations for effective thermal expansions, stored energy and average thermoelastic strains inside the components can be found. By this means in the framework of the effective field hypothesis one obtains the generalization of the classical Rosen and Hashin's (1970) formula, which is exact for two-component composites. The proposed theory is applied to composites reinforced with particles with thin, along inclusion surface inhomogeneous coatings of the inclusions. For a single coated inclusion the micromechanical approach is based on the Green's function technique as well as on the interfacial Hill operators.

2. Preliminaries

2.1. Basic equations

The paper discussed a certain representative mesodomain w with a characteristic function W containing a set $X = (v_i)$ of inclusions v_i with characteristic functions $V_i (i = 1, 2, \ldots)$. At first no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the inclusions. The inclusions are determined as the component $v^{(1)}$ having identical mechanical and geometrical properties. The local strain tensor $\boldsymbol{\varepsilon}$ is related to the displacement $\mathbf{u}$ via the linearized strain–displacement equation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^{\mathrm{T}}\right]. \tag{2.1}$$

Here $\otimes$ denotes tensor product and $(^\bullet)^{\mathrm{T}}$ denotes matrix transposition. The stress tensor $\boldsymbol{\sigma}$, satisfies the equilibrium equation (no body forces acting):

$$\nabla\boldsymbol{\sigma} = \mathbf{0}. \tag{2.2}$$

Stresses and strains are related to each other via the constitutive equation

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}) + \boldsymbol{\alpha}(\mathbf{x}) \quad \text{or } \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{M}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x}) + \boldsymbol{\beta}(\mathbf{x}). \tag{2.3}$$

$\mathbf{L}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x}) \equiv \mathbf{L}(\mathbf{x})^{-1}$ are known phase stiffness and compliance fourth-order tensors and the common notation for tensor products has been employed: $\mathbf{L}\varepsilon = [L_{ijkl}\boldsymbol{\varepsilon}_{kl}]$, $\boldsymbol{\sigma\varepsilon} = [\boldsymbol{\sigma}_{ij}\varepsilon_{ij}]$, $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = [\alpha_{ij}\beta_{kl}]$. $\boldsymbol{\beta}(\mathbf{x})$ and $\boldsymbol{\alpha}(\mathbf{x}) \equiv -\mathbf{L}(\mathbf{x})\boldsymbol{\beta}(\mathbf{x})$ are second-order tensors of local eigenstrains and eigenstresses, respectively (frequently called transformation fields) which may arise by thermal expansion, phase transformation, twinning and other changes of shape or volume of the material. All tensors $\mathbf{f}$ ($\mathbf{f} = \mathbf{L}$, $\mathbf{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$) of material properties are decomposed as, respectively, $\mathbf{f} \equiv \mathbf{f}^{(0)} + \mathbf{f}_1(\mathbf{x})$. $\mathbf{f}$ is assumed to be constant in the matrix $v^{(0)} = w \backslash v^{(1)}$ and is an inhomogeneous function inside the inclusions, respectively,

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{f}^{(0)} & \text{for } \mathbf{x} \in v^{(0)} \\ \mathbf{f}^{(0)} + \mathbf{f}_1^{(1)}(\mathbf{x}) & \text{for } \mathbf{x} \in v_i \subset v^{(1)}. \end{cases} \tag{2.4}$$

Particularly, for coated inclusions this means that $\mathbf{f}_1^{(1)}(\mathbf{x})$ describes the change of properties between the core and the coating of the inclusion v_i. Here and in the following the upper index (k) $(k = 0, 1)$ numbers the components and the lower index i numbers the individual inclusions; $v^{(1)} \equiv \cup v_i (i = 1, 2, \ldots)$.

We assume that the phases are perfectly bonded, so that the displacements and the traction components of the stresses are continuous across the interphase boundaries. We take uniform traction boundary conditions

$$\boldsymbol{\sigma}^0\mathbf{n}(\mathbf{x}) = \mathbf{T}(\mathbf{x}), \quad \mathbf{x} \in \partial w, \tag{2.5}$$

where $\mathbf{T}(\mathbf{x})$ is the traction vector at the external boundary ∂w of the mesodomain w, $\mathbf{n}$ is its unit outward normal and $\boldsymbol{\sigma}^0$, the mesoscopic stress tensor, is a given constant symmetric tensor.

2.2. Statistical description of the composite structure

It is assumed that the representative mesodomain w contains a statistically large number of inclusions v_i; all the random quantities under discussion are described by statistically homogeneous ergodic random fields and thereby, the ensemble averaging could be replaced by volume averaging

$$\langle(\cdot)\rangle = \bar{w}^{-1}\int(\cdot)W(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad \langle(\cdot)\rangle^{(k)} = \left[\bar{v}^{(k)}\right]^{-1}(\cdot)V^{(k)}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad (k = 0, 1). \tag{2.6}$$

The bar appearing above the region represents its measure, e.g. $\bar{v} \equiv \text{mes } v$. $V^{(k)}$ is the characteristic functions of $v^{(k)}$. The average over an individual inclusion $v_i \subset v^{(1)}$ $(i = 1, 2, \ldots)$: $\langle(^\bullet)\rangle_i = \langle(^\bullet)\rangle^{(1)}$.

For the description of the random structure of a composite material let us introduce a conditional probability density $\varphi(v_m|\mathbf{x}_m, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, which is a probability density to find the m-th inclusion with the center $\mathbf{x}_m$ in the domain v_m with fixed inclusions $v_1, \ldots, v_n$ with the centers $\mathbf{x}_1, \ldots, \mathbf{x}_n$. The notation $\varphi(v_m|\mathbf{x}_m; \mathbf{x}_1, \ldots, \mathbf{x}_n)$ denotes the case $\mathbf{x}_m \neq \mathbf{x}_1, \ldots, \mathbf{x}_n$. Of course, $\varphi(v_m|\mathbf{x}_m; \mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ for values of $\mathbf{x}_m$ lying inside the 'included volumes' $\cup v_{0i} (i = 1, \ldots, n)$, where $v_{0i} \supset v_i$ with characteristic functions V_{0i} (since inclusions cannot overlap), and $\varphi(v_m|\mathbf{x}_m; \mathbf{x}_1, \ldots, \mathbf{x}_n) \to \varphi(v_m)$ at $|\mathbf{x}_i - \mathbf{x}_m| \to \infty$, $i = 1, \ldots, n$ (since no long-range order is assumed). Since the coating is defined here as being a part of the inclusion, no distinction between coated and uncoated inclusion is necessary. $\varphi(v_m)$ is a number density $n^{(1)}$ of the inclusions; $c^{(k)}$ is the concentration, i.e. volume fraction, of the component $v^{(k)}$: $c^{(k)} = \langle V^{(k)}\rangle$; $c^{(1)} = \bar{v}_m n^{(1)}$, $c^{(0)} = 1 - c^{(1)}$ $(k = 0, 1; m = 1, 2, \ldots)$. Only if the pair distribution function $g(\mathbf{x}_m - \mathbf{x}_i) =$

$\varphi(v_m|\mathbf{x}_m;\mathbf{x}_i)/n^{(1)}$ depends on $|\mathbf{x}_m-\mathbf{x}_i|$ it is called the radial distribution function. Below the notation $\langle(^\bullet)(\mathbf{x})|v_1,\ \mathbf{x}_1;\ldots;v_m,\ \mathbf{x}_m\rangle$ will be used for the conditional average taken for the ensemble of a statistically homogeneous ergodic field $X=(v_i)$, on the condition that there are inclusions $v_1,\ldots,v_m$ at the points $\mathbf{x}_1,\ldots,\mathbf{x}_m$ and $\mathbf{x}_1\neq\ldots\neq\mathbf{x}_m$. The notation $\langle(^\bullet)(\mathbf{y})|v_1,\ \mathbf{x}_1;\ldots;v_m,\ \mathbf{x}_m\rangle$ is used for the case $\mathbf{y}\notin v_1,\ldots,v_m$. The notation for the conditional probability density $\varphi(v_p|\mathbf{x}_p;\ldots;\mathbf{x}_0)$ is considered under the condition that the inclusions $v_p,\ldots$ are located in the points $\mathbf{x}_p,\ldots$, whereas the matrix position is denoted by $\mathbf{x}_0$.

2.3. Overall thermoelastic properties

We now summarize the principal formulae of thermostatics in a form which is appropriate to our intended application to composites (see e.g. Laws, 1973; Kreher and Pompe, 1989; Dvorak and Benveniste, 1992). For the estimation of effective properties the average values of strain and stress concentration tensors are needed, i.e. no detailed distinction is required between coating and core of the inclusions.

Let us decompose the overall field as

$$\boldsymbol{\sigma}(\mathbf{x})=\boldsymbol{\sigma}^{\mathrm{I}}(\mathbf{x})+\boldsymbol{\sigma}^{\mathrm{II}}(\mathbf{x}),\quad \boldsymbol{\varepsilon}(\mathbf{x})=\boldsymbol{\varepsilon}^{\mathrm{I}}(\mathbf{x})+\boldsymbol{\varepsilon}^{\mathrm{II}}(\mathbf{x}),\tag{2.7}$$

with the sources

$$\boldsymbol{\alpha}^{\mathrm{I}}(\mathbf{x})=\boldsymbol{\beta}^{\mathrm{I}}(\mathbf{x})=0,\quad \mathbf{T}^{\mathrm{I}}(\mathbf{x})=\mathbf{T}(\mathbf{x}),\tag{2.8}$$

$$\boldsymbol{\alpha}^{\mathrm{II}}(\mathbf{x})=\boldsymbol{\alpha}(\mathbf{x}),\quad \boldsymbol{\beta}^{\mathrm{II}}(\mathbf{x})=\boldsymbol{\beta}(\mathbf{x}),\quad \mathbf{T}^{\mathrm{II}}(\mathbf{x})=0.\tag{2.9}$$

The contribution to the local field (2.3) from a purely mechanical load is

$$\boldsymbol{\varepsilon}^{\mathrm{I}}(\mathbf{x})=\mathbf{A}^*(\mathbf{x})\langle\boldsymbol{\varepsilon}\rangle,\quad \boldsymbol{\sigma}^{\mathrm{I}}(\mathbf{x})=\mathbf{B}^*(\mathbf{x})\langle\boldsymbol{\sigma}\rangle,\tag{2.10}$$

where $\mathbf{A}^*$ and $\mathbf{B}^*$ are fourth-order tensors; their phase volume averages, i.e. average over $v^{(k)}$ in a representative volume $\mathbf{A}^*_k\equiv\langle\mathbf{A}^*\rangle^{(k)}$ and $\mathbf{B}^*_k\equiv\langle\mathbf{B}^*\rangle^{(k)}$ ($k=0,\ 1$) are called the mechanical strain and stress concentration tensors; it is necessary that $\langle\mathbf{A}^*\rangle=\mathbf{I}$, $\langle\mathbf{B}^*\rangle=\mathbf{I}$, where $\mathbf{I}$ is the unit fourth-order tensor.

The overall constitutive relations are written as

$$\langle\boldsymbol{\sigma}\rangle\equiv\mathbf{L}^*\langle\boldsymbol{\varepsilon}\rangle+\boldsymbol{\alpha}^*,\quad \langle\boldsymbol{\varepsilon}\rangle=\mathbf{M}^*\langle\boldsymbol{\sigma}\rangle+\boldsymbol{\beta}^*.\tag{2.11}$$

Then effective parameters $\mathbf{M}^*=(\mathbf{L}^*)^{-1}$, $\boldsymbol{\beta}^*=-\mathbf{M}^*\boldsymbol{\alpha}^*$ are found from the solution of the elastic problem (2.10)

$$\mathbf{M}^*=\langle\mathbf{M}\mathbf{B}^*\rangle,\quad \mathbf{L}^*=\langle\mathbf{L}\mathbf{A}^*\rangle,\tag{2.12}$$

$$\boldsymbol{\alpha}^*=\langle\mathbf{A}^{*\mathrm{T}}\boldsymbol{\alpha}\rangle,\quad \boldsymbol{\beta}^*=\langle\mathbf{B}^{*\mathrm{T}}\boldsymbol{\beta}\rangle.\tag{2.13}$$

According to the Hill condition (Hill, 1963), the uniform constraint condition (2.5) has the following properties

$$\langle\boldsymbol{\sigma}\rangle=\boldsymbol{\sigma}^0,\quad \langle\boldsymbol{\varepsilon}\rangle=\bar{w}^{-1}\int\frac{1}{2}(\mathbf{u}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{u})\,\mathrm{d}s,\tag{2.14}$$

$$\langle\boldsymbol{\varepsilon}\boldsymbol{\sigma}\rangle=\langle\boldsymbol{\varepsilon}\rangle\langle\boldsymbol{\sigma}\rangle.\tag{2.15}$$

For simplicity the argument ($\mathbf{x}$) will frequently be dropped. Eqn (2.14) means that the average stress $\langle\boldsymbol{\sigma}\rangle$ is precisely $\boldsymbol{\sigma}^0$ and that the average strain $\langle\boldsymbol{\varepsilon}\rangle$ can be 'measured' in terms of the boundary displacements. The Hill condition (2.15) holds for any compatible mean strain field $\boldsymbol{\varepsilon}(\mathbf{x})$ (2.1) and equilibrium stress field $\boldsymbol{\sigma}(\mathbf{x})$ (2.2) not necessarily related to each other by a specific stress–strain relation.

By using the Hill condition (2.15) we find four scalar equations

$$\langle\boldsymbol{\varepsilon}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}\rangle = \langle\boldsymbol{\sigma}\rangle\mathbf{M}^*\langle\boldsymbol{\sigma}\rangle, \quad \langle\boldsymbol{\varepsilon}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{I}}\rangle = \boldsymbol{\beta}^*\langle\boldsymbol{\sigma}\rangle, \tag{2.16}$$

$$\langle\boldsymbol{\varepsilon}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}}\rangle = 0, \quad \langle\boldsymbol{\varepsilon}^{\mathrm{II}}\boldsymbol{\sigma}^{\mathrm{II}}\rangle = 0. \tag{2.17}$$

According to (2.17) the average strain energy density of the loaded material is given by

$$U^* \equiv \frac{1}{2}\langle\boldsymbol{\sigma}\mathrm{M}\boldsymbol{\sigma}\rangle = \frac{1}{2}\langle\boldsymbol{\varepsilon}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{I}}\rangle + \frac{1}{2}\langle\left(\boldsymbol{\varepsilon}^{\mathrm{II}} - \boldsymbol{\beta}\right)\boldsymbol{\sigma}^{\mathrm{II}}\rangle. \tag{2.18}$$

Then the overall strain energy density U^* is the sum of the strain energy density $U^{*\mathrm{I}}$ due to the applied load and $U^{*\mathrm{II}}$ stored by the transformation stress field,

$$U^* = U^{*\mathrm{I}} + U^{*\mathrm{II}}, \quad U^{*\mathrm{I}} = \frac{1}{2}\langle\boldsymbol{\sigma}\rangle\mathbf{M}^*\langle\boldsymbol{\sigma}\rangle, \quad U^{*\mathrm{II}} = -\frac{1}{2}\langle\boldsymbol{\beta}\boldsymbol{\sigma}^{\mathrm{II}}\rangle. \tag{2.19}$$

The stores energy $U^{*\mathrm{II}}$, like $\boldsymbol{\beta}^*$, depends only on the fields $\mathbf{L}(\mathbf{x})$ and $\boldsymbol{\beta}(\mathbf{x})$ and acts like effective material constants, which can be observed macroscopically. $\boldsymbol{\beta}^* = \langle\boldsymbol{\varepsilon}^{\mathrm{II}}\rangle$ is the tensor of effective stress-free strains and $U^{*\mathrm{II}}$ can be determined experimentally by measuring the specific heats of the composite (see e.g. Christensen, 1979; Buryachenko and Shermergor, 1995).

3. General integral equation and effective field hypothesis

From Eqns (2.1)–(2.4) a general integral equation for $\boldsymbol{\varepsilon}$ can be derived. Substituting (2.1) and (2.3) into the equilibrium equation (2.2), we obtain a differential equation with respect to the displacement $\mathbf{u}$. By rearranging the latter equation into an integral one and transforming it by a method developed earlier (see e.g. Levin, 1976; Kröner, 1977; Willis, 1982; Buryachenko and Kreher, 1995), we obtain

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^0 + \int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\left\{\mathbf{M}_1(\mathbf{y}) + \boldsymbol{\beta}_1(\mathbf{y}) - \left[\langle\mathbf{M}_1\boldsymbol{\sigma}\rangle + \langle\boldsymbol{\beta}_1\rangle\right]\right\}\,\mathrm{d}\mathbf{y}, \tag{3.1}$$

where

$$\mathbf{M}_1(\mathbf{y}) = \begin{cases} \mathbf{0} & \text{for } \mathbf{y} \in v^{(\mathbf{0})} \\ \mathbf{M}^{(1)}(\mathbf{y}) - \mathbf{M}^{(\mathbf{0})} & \text{for } \mathbf{y} \in v_i \subset v^{(1)} \end{cases}, \tag{3.2}$$

is the jump of the compliance $\mathbf{M}^{(1)}$ ($\mathbf{y}$) of the component $v^{(1)}$ with respect to the matrix $v^{(0)}$, $\mathbf{M}^{(k)} \equiv [\mathbf{L}^{(k)}]^{-1}$ ($k = 0, 1$). By the function $\mathbf{M}^{(1)}$ ($\mathbf{y}$) the variation of the material properties within coated inclusions is taken into account. The integral operator kernel

$$\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) \equiv -\mathbf{L}^{(0)}\left[\mathbf{I}\delta(\mathbf{x}-\mathbf{y}) + \nabla\nabla\mathbf{G}(\mathbf{x}-\mathbf{y})\mathbf{L}^{(0)}\right], \tag{3.3}$$

is defined by the Green tensor $\mathbf{G}$ of the Lamé equation of a homogeneous medium with an elastic modulus tensor $\mathbf{L}^{(0)}$

$$\nabla\left\{\mathbf{L}^{(0)}\frac{1}{2}\left[\nabla\otimes\mathbf{G}(\mathbf{x})+(\nabla\otimes\mathbf{G}(\mathbf{x}))^{\mathrm{T}}\right]\right\}=-\boldsymbol{\delta}\delta(\mathbf{x}), \tag{3.4}$$

$\delta(\mathbf{x})$ is the Dirac delta function, $\boldsymbol{\delta}$ and $\mathbf{I}$ are the unit second-order and fourth-order tensors, respectively.

Let us consider some conditional statistical averages of the general integral equation (3.1) leading to an infinite system of integral equations ($n = 1, 2, \ldots$)

$$\langle\boldsymbol{\sigma}(\mathbf{x})|v_1,\mathbf{x}_1;\ldots;v_n,\mathbf{x}_n\rangle-\sum_{i=1}^{n}\int\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\langle V_i(\mathbf{y})\left[\mathbf{M}_1(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y})+\boldsymbol{\beta}_1(\mathbf{y})\right]|v_1,\mathbf{x}_1;\ldots;v_n,\mathbf{x}_n\rangle\,\mathrm{d}\mathbf{y}$$

$$=\boldsymbol{\sigma}^0+\int\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\left\{\langle\mathbf{M}_1(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y})+\boldsymbol{\beta}_1(\mathbf{y})|;v_1,\mathbf{x}_1;\ldots;v_n,\mathbf{x}_n\rangle-\left[\langle\mathbf{M}_1\boldsymbol{\sigma}\rangle+\langle\boldsymbol{\beta}_1\rangle\right]\right\}\mathrm{d}\mathbf{y}, \tag{3.5}$$

where $\mathbf{x}\in v_1,\ldots,v_n$ in the n-th line of the system (see Buryachenko and Kreher, 1995).

Now we define the effective field $\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\ldots,}\ n(\mathbf{x}\in v_1,\ldots,v_n)$ as a stress field in which the chosen fixed inclusions $v_1,\ldots,v_n$ are embedded. This effective field is a random function of all the other positions of the surrounding inhomogeneities and the average of $\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\ldots,}\ n$ over a random realization of these inclusions is equal to the right-hand-side of the n-th line of the system (3.5)

$$\langle\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\ldots,\,n}\rangle=\boldsymbol{\sigma}^0+\int\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\left\{\langle\mathbf{M}_1(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y})+\boldsymbol{\beta}_1(\mathbf{y})|;v_1,\mathbf{x}_1;\ldots;v_n,\mathbf{x}_n\rangle-\left[\langle\mathbf{M}_1\boldsymbol{\sigma}\rangle+\langle\boldsymbol{\beta}_1\rangle\right]\right\}\mathrm{d}\mathbf{y}, \tag{3.6}$$

where ($\mathbf{x}\in v_i$, $i=1,2,\ldots,n$). Consequently, each inclusion $v_i(i=1,\ldots,n)$ of the chosen fixed set is in a random (generally speaking nonhomogeneous) field

$$\bar{\boldsymbol{\sigma}}(\mathbf{x})=\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\ldots,\,n}+\sum_{j\neq i}\int\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})V_j(\mathbf{y})\left[\mathbf{M}_1(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y})+\boldsymbol{\beta}_1(\mathbf{y})\right]\mathrm{d}\mathbf{y}, \tag{3.7}$$

($\mathbf{x}\in v_i$, $j\neq i; i,\ j=1,\ 2,\ldots,\ n$) which is the superposition of the effective field $\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\ldots n}$ and the distribution caused by the other inclusions of the considered set.

In order to simplify the exact system (3.5) we now apply the main hypothesis of many micromechanical methods, the so-called effective field hypothesis.

H1: Each inclusion v_i has an ellipsoidal shape and is embedded in the field $\tilde{\boldsymbol{\sigma}}_i(\mathbf{x})$ which is homogeneous over the inclusion v_i. The perturbation introduced by the inclusion v_i in the point $\mathbf{y}\notin v_i$ is defined by the relation

$$\int\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})V_i(\mathbf{x})\left[\mathbf{M}_1\ (\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})+\boldsymbol{\beta}_1(\mathbf{x})\right]\mathrm{d}\mathbf{x}=\bar{v}_i\mathbf{T}_i(\mathbf{y}-\mathbf{x}_i)\langle\mathbf{M}_1(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})+\boldsymbol{\beta}_1(\mathbf{x})\rangle_{(i)} \tag{3.8}$$

where $\langle(\cdot)\rangle_{(i)}$ is an average over the volume of the inclusion v_i (but not over the ensemble) and

$$\mathbf{T}_i(\mathbf{y}-\mathbf{x}_i)=(\bar{v}_i)^{-1}\int\boldsymbol{\Gamma}(\mathbf{y}-\mathbf{x})V_i(\mathbf{x})\,\mathrm{d}\mathbf{x},\quad \mathbf{y}\notin v_i. \tag{3.9}$$

In analogy to Buryachenko and Parton (1992c) and in view of linearity of the problem there exist constant fourth- and second-rank tensors $\mathbf{B}$ and $\mathbf{C}$, respectively, such that

$$\langle\boldsymbol{\sigma}(\mathbf{x})\rangle_i=\mathbf{B}\langle\bar{\boldsymbol{\sigma}}(\mathbf{x})\rangle_i+\mathbf{C},\quad \mathbf{x}\in v_i,$$

$$\bar{v}_i\langle\mathbf{M}_1(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})+\boldsymbol{\beta}_1(\mathbf{x})\rangle_i=\mathbf{R}\langle\bar{\boldsymbol{\sigma}}(\mathbf{x})\rangle_i+\mathbf{F}, \tag{3.10}$$

where the tensors **R** and **F** are found by the use of the Eshelby theorem (Eshelby, 1961)

$$\mathbf{R} = \bar{v}_i \mathbf{Q}^{-1}(\mathbf{I} - \mathbf{B}), \quad \mathbf{F} = -\bar{v}_i \mathbf{Q}^{-1}\mathbf{C}. \tag{3.11}$$

The tensor **Q** is associated with the well-known Eshelby tensor **S** by

$$\mathbf{S} = \mathbf{I} - \mathbf{M}^{(0)}\mathbf{Q}, \quad \mathbf{Q} \equiv -\langle \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}) \rangle_i = \text{ const} \quad (\mathbf{x}, \mathbf{y} \in v_i). \tag{3.12}$$

Of course, in practice the tensors **B** and **C** are found from the thermoelastic problem of a single inclusion v_i in the infinite matrix, when $c^{(1)} = 0$ and $\bar{\boldsymbol{\sigma}}_i(\mathbf{x}) \equiv \boldsymbol{\sigma}^0$. This problem is connected with the calculation of the inhomogeneous tensors $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$ by either analytical or numerical methods, such that for $\mathbf{x} \in v_i$ the following holds:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\boldsymbol{\sigma}^0 + \mathbf{C}(\mathbf{x}). \tag{3.13}$$

$$\mathbf{B} = \langle \mathbf{B}(\mathbf{x}) \rangle_i, \quad \mathbf{C} = \langle \mathbf{C}(\mathbf{x}) \rangle_i,$$

$$\mathbf{R} = \bar{v}_i \langle \mathbf{M}_1(\mathbf{x})\mathbf{B}(\mathbf{x}) \rangle_i, \quad \mathbf{F} = \bar{v}_i \langle \mathbf{M}_1\ (\mathbf{x})\mathbf{C}(\mathbf{x}) + \boldsymbol{\beta}_1(\mathbf{x}) \rangle_i. \tag{3.14}$$

In Section 5 we consider an analytical method for the calculation of the tensors $\mathbf{B}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$ for ellipsoidal inclusions with a thin coating. Other analytical methods for the analysis of coated ellipsoidal inclusions are mentioned in the Introduction. In the general case the estimation of the tensors $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$ is a particular problem of the transfomation field analysis method by Dvorak and Benveniste (1992) and is not discussed in more detail in this paper.

For the particular case of the homogeneous ellipsoidal domain v_i (uncoated inclusions) with $\mathbf{M}_1(\mathbf{x}) = \mathbf{M}_1^{(1)} \equiv$ const, $\boldsymbol{\beta}_1(\mathbf{x}) = \boldsymbol{\beta}_1^{(1)} \equiv$ const, we have

$$\mathbf{B} = \left(\mathbf{I} + \mathbf{Q}\mathbf{M}_1^{(1)}\right)^{-1}, \ \mathbf{C} = -\mathbf{B}\mathbf{Q}\boldsymbol{\beta}_1^{(1)}, \tag{3.15}$$

$$\mathbf{R} = \bar{v}_1 \mathbf{M}_1^{(1)}\left(\mathbf{I} + \mathbf{Q}\mathbf{M}_1^{(1)}\right)^{-1}, \quad \mathbf{F} = \bar{v}_1\left(\mathbf{I} + \mathbf{M}_1^{(1)}\mathbf{Q}\right)^{-1}\boldsymbol{\beta}_1^{(1)}. \tag{3.16}$$

By comparison of relation (3.10) with (3.15) we see that the average thermoelastic response (i.e. the tensors **B**, **C**, **R**, **F**) of any coated inclusion is the same as that of some fictitious ellipsoidal homogeneous, i.e. uncoated, inclusion with thermoelastic parameters

$$\mathbf{M}_1^{f(1)} = \mathbf{Q}^{-1}(\mathbf{B}^{-1} - \mathbf{I}), \quad \boldsymbol{\beta}_1^{f(1)} = -\mathbf{Q}\mathbf{B}^{-1}\mathbf{C}, \tag{3.17}$$

which also can be expressed in terms of the tensors **R** and **F**

$$\mathbf{M}_1^{f(1)} = \mathbf{R}(\mathbf{I}\bar{v}_i - \mathbf{Q}\mathbf{R})^{-1}, \quad \boldsymbol{\beta}_1^{f(1)} = \bar{v}_i^{-1}\left(\mathbf{M}_1^{f(1)}\mathbf{Q} + \mathbf{I}\right)\mathbf{F}. \tag{3.18}$$

The parameters (3.17) and (3.18) of fictitious ellipsoidal inclusions are simply a notational convenience. No restrictions are imposed on the microtopology of the coated inclusions as well as on the inhomogeneity of the stress state in the coated inclusions.

In analogy to Willis (1982) we define

$$\boldsymbol{\eta}(\mathbf{x}) \equiv \mathbf{R}\bar{\boldsymbol{\sigma}}(\mathbf{x}) + \mathbf{F}, \quad \boldsymbol{\eta}^0 \equiv \mathbf{R}\boldsymbol{\sigma}^0 + \mathbf{F} \quad (\mathbf{x} \in v_i) \tag{3.19}$$

These quantities are called strain polarization tensors $\boldsymbol{\eta}(\mathbf{x})$ and $\boldsymbol{\eta}^0$.

Averaging (3.5) over the volume of the considered inclusion v_i and using the hypothesis **H1** (3.8) with (3.19) leads to

$$\langle\boldsymbol{\eta}(\mathbf{x})|v_1,\mathbf{x};\dots;v_n,\mathbf{x}_n\rangle_i-\sum_{j\neq i}^{n}\mathbf{R}\mathbf{T}_{ij}(\mathbf{x}_i-\mathbf{x}_j)\langle\boldsymbol{\eta}(\mathbf{y})|v_1,\mathbf{x}_1;\dots;v_n,\mathbf{x}_n\rangle_j$$

$$=\boldsymbol{\eta}^0+\mathbf{R}\int\big[\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q)\langle\boldsymbol{\eta}(\mathbf{y})|;v_1,\mathbf{x}_1;\dots;v_n,\mathbf{x}_n\rangle_q$$

$$\times\varphi\big(v_q,\mathbf{x}_q|;v_1,\mathbf{x}_1;\dots v_n,\mathbf{x}_n\big)-\mathbf{T}_i(\mathbf{x}_i-\mathbf{x}_q)\langle\boldsymbol{\eta}\rangle n^{(1)}\big]\,\mathrm{d}\mathbf{x}_q \tag{3.20}$$

($n=1,2,\dots;\ i,j=1,\dots,n$), where the tensors

$$\mathbf{T}_{ij}(\mathbf{x}-\mathbf{x}_j)=\big(\bar{v}_i\bar{v}_j\big)^{-1}\int\int\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})V_i(\mathbf{x})V_j(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \tag{3.21}$$

and the tensors $\mathbf{T}_i(\mathbf{y}-\mathbf{x}_i)$ (3.9) have an analytical representation for spherical inclusions in an isotropic matrix (see e.g. Buryachenko and Rammerstorfer, 1997), regardless of whether these inclusions are coated or uncoated.

4. Effective properties

4.1. Average stresses inside the components

The system (3.20) has principally the same structure as the system for the pure elastic problem (with $\mathbf{F}\equiv\mathbf{0}$). Therefore, we can apply the traditional analysis procedure of purely elastic composites and represent $\langle\boldsymbol{\eta}\rangle^{(1)}$ as a linear function of the external field $\boldsymbol{\eta}^0$

$$\langle\boldsymbol{\eta}\rangle^{(1)}=\mathbf{Y}\boldsymbol{\eta}^0 \tag{4.1}$$

and therefore,

$$\mathbf{R}\langle\bar{\boldsymbol{\sigma}}\rangle^{(1)}+\mathbf{F}=\mathbf{Y}\big(\mathbf{R}\boldsymbol{\sigma}^0+\mathbf{F}\big). \tag{4.2}$$

The comparison of (3.20) with (4.1) leads to the fact that $\mathbf{Y}$ only depends on the tensors $\mathbf{R}$, $\mathbf{T}_{ij}$ and $\mathbf{T}_j$. The tensor $\mathbf{Y}$ is determined by the purely elastic action (with $\mathbf{F}\equiv\mathbf{0}$) of the surrounding inclusions on the separated one. For a dilute concentration of the inclusions, i.e. $c^{(1)}\to 0$, we have $\mathbf{Y}\to\mathbf{I}$. The actual form of the tensor $\mathbf{Y}$, used in the analysis as an approximation, depends on additional assumptions for closing of the infinite system (3.20). In particular, for purely elastic composites (with $\boldsymbol{\beta}_1\equiv\mathbf{0}$) with fictitious homogeneous inclusions (3.16) and (3.17) such relations are represented in Appendix A for commonly applied methods of micromechanics, i.e. effective medium method by Kröner (1961) and by Hill (1965), Mori–Tanaka method, MEFM.

This means, taking into account eqns (3.10), (3.19) and (4.1), from the solution of the purely elastic problem for the composite, we can calculate the average stresses inside the inclusions by

$$\langle\boldsymbol{\sigma}\rangle^{(1)}=\mathbf{B}\mathbf{R}^{-1}\big[\mathbf{Y}\big(\mathbf{R}\boldsymbol{\sigma}^0+\mathbf{F}\big)-\mathbf{F}\big]+\mathbf{C}. \tag{4.3}$$

The mean matrix stresses follow simply from the condition $\langle\boldsymbol{\sigma}\rangle = \boldsymbol{\sigma}^0$ (2.14); and

$$\langle\boldsymbol{\sigma}\rangle^{(0)} = \left(c^{(0)}\right)^{-1}\left[\boldsymbol{\sigma}^0 - c^{(1)}(\mathbf{I}-\mathbf{B})^{-1}C - c^{(1)}\mathbf{B}\mathbf{R}^{-1}\mathbf{Y}(\mathbf{R}\boldsymbol{\sigma}+\mathbf{F})\right]. \tag{4.4}$$

The local stresses inside the inclusion, i.e. in the core and in the coating, respectively, are found by

$$\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}) = \mathbf{C}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\left[\mathbf{Y}\left(\mathbf{R}\boldsymbol{\sigma}^0+\mathbf{F}\right)-\mathbf{F}\right], \tag{4.5}$$

where $\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})$ means the average of the local stress state at $\mathbf{x}\in v_i \subset v^1$ over an ensemble realization of surrounding inclusions (but not over the volume v_i of a particular inclusion, in contrast to $\langle\boldsymbol{\sigma}\rangle^{(1)}$).

Comparing (2.10) with (4.5) leads to the relation for the average local stress concentration tensor inside the inclusions

$$\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}) = \langle\mathbf{B}^*\rangle_i(\mathbf{x})\boldsymbol{\sigma}^0 + \left[\langle\mathbf{B}^*\rangle_i(\mathbf{x}) - \mathbf{B}(\mathbf{x})\right]\mathbf{R}^{-1}\mathbf{F} + \mathbf{C}(\mathbf{x}), \tag{4.6}$$

where $\langle\mathbf{B}^*\rangle_i(\mathbf{x}) \equiv \mathbf{B}(\mathbf{x})\mathbf{D}$, $(\mathbf{x}\in v_i)$ and the tensor $\mathbf{D}\equiv\mathbf{R}^{-1}\mathbf{Y}\mathbf{R}$ has the simple physical meaning of the action [at $\boldsymbol{\beta}(\mathbf{x})\equiv\mathbf{0}$] of the surrounding inclusions on the separated one: $\langle\bar{\boldsymbol{\sigma}}\rangle_i = \mathbf{D}\boldsymbol{\sigma}^0$.

Thus, the average of local thermal stresses $\langle\boldsymbol{\sigma}^{\mathrm{II}}\rangle_i(\mathbf{x})$ (2.7) over the ensemble is defined by the purely elastic solution for the composite medium [the tensor $\langle\mathbf{B}^*\rangle_i(\mathbf{x})$] as well as by the thermoelastic solution for a single inclusion in an infinite matrix [the tensors $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$, $\mathbf{R}$, $\mathbf{F}$]. For two-component composites with identical homogeneous inclusions of any shape Benveniste and Dvorak (1990) obtained an exact relation for non-averaged local thermal stresses

$$\boldsymbol{\sigma}^{\mathrm{II}}(\mathbf{x}) = \left[\mathbf{B}^*(\mathbf{x}) - \mathbf{I}\right]\left[\mathbf{M}_1^{(1)}\right]^{-1}\boldsymbol{\beta}_1^{(1)},$$

from which one can derive eqn (4.6) in the case of ellipsoidal inclusions.

The matrix stresses in the immediate vicinity of the inclusions v_i, denoted by $\boldsymbol{\sigma}_i^-(\mathbf{n})$, are given by the formula (see Appendix B)

$$\boldsymbol{\sigma}_i^-(\mathbf{n}) = \boldsymbol{\sigma}_i^+(\mathbf{x}) + \boldsymbol{\Gamma}(\mathbf{n})\left[\mathbf{M}_1^{(1)}(\mathbf{x})\boldsymbol{\sigma}_i^+(\mathbf{x}) + \boldsymbol{\beta}_1^{(1)}(\mathbf{x})\right], \tag{4.7}$$

where $\boldsymbol{\sigma}_i^-(\mathbf{n})$ and $\boldsymbol{\sigma}_i^+(\mathbf{x})$ are the limiting stresses outside and inside, respectively, near the inclusion boundary ∂v_i: $\boldsymbol{\sigma}_i^-(\mathbf{n}) = \lim\boldsymbol{\sigma}(\mathbf{y})$; $\boldsymbol{\sigma}_i^+ = \lim\boldsymbol{\sigma}(\mathbf{z})$, $\mathbf{y}\to\mathbf{x}$, $\mathbf{z}\to\mathbf{x}$, $\mathbf{y}\in v_0$, $\mathbf{z}\in v_i$, $\mathbf{x}\in\partial v_i$; $\mathbf{n}$ is the unit outward normal vector on ∂v_i. The relation (4.6) is correct for any shape of the inclusion v_i. The tensor $\boldsymbol{\Gamma}(\mathbf{n})$ depends only on the elastic properties of the matrix material $\mathbf{M}^{(0)}$ and on the direction of the normal $\mathbf{n}$; the expression for $\boldsymbol{\Gamma}(\mathbf{n})$ is presented in Appendix B.

The eqns (4.5) and (4.6) allow the estimation of the ensemble average of the matrix stresses in the vicinity of the inclusions near a point $\mathbf{x}\in v_i$

$$\langle\boldsymbol{\sigma}^-(\mathbf{n})\rangle_x = \left[\mathbf{I} + \boldsymbol{\Gamma}(\mathbf{n})\mathbf{M}_1^{(1)}(\mathbf{x})\right]\left\{\mathbf{C}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\left[\mathbf{Y}\left(\mathbf{R}\boldsymbol{\sigma}^0+\mathbf{F}\right)-\mathbf{F}\right]\right\} + \boldsymbol{\Gamma}(\mathbf{n})\boldsymbol{\beta}_1^{(1)}(\mathbf{x}). \tag{4.8}$$

4.2. Overall properties and correlations between them

For the estimation of the effective compliance we use the relations (2.12) and (4.5) to obtain

$$\mathbf{M}^* = \mathbf{M}^{(0)} + \mathbf{Y}\mathbf{R}n^{(1)}. \tag{4.9}$$

Taking the equality $\boldsymbol{\beta}^* = \boldsymbol{\beta}^{(0)} + \langle\mathbf{M}_1\boldsymbol{\sigma}^{\mathrm{II}} + \boldsymbol{\beta}_1\rangle$ (see e.g. Buryachenko and Lipanov, 1986a) into account we find the coefficient of thermal expansion from the relations (3.10) and (4.2):

$$\boldsymbol{\beta}^* = \boldsymbol{\beta}^{(0)} + \mathbf{YF}n^{(1)}. \tag{4.10}$$

Eqns (2.19) and (4.5) yield the following relation for the stored energy density:

$$U^{*\mathrm{II}} = -\frac{1}{2}\langle\boldsymbol{\beta}_1(\mathbf{x})\mathbf{B}(\mathbf{x})\rangle\mathbf{R}^{-1}(\mathbf{Y}-\mathbf{I})\mathbf{F} - \frac{1}{2}\langle\boldsymbol{\beta}_1(\mathbf{x})\mathbf{C}(\mathbf{x})\rangle. \tag{4.11}$$

Interestingly, all three effective quantities $\mathbf{M}^*$, $\boldsymbol{\beta}^*$, $U^{*\mathrm{II}}$ can be estimated by the use of a unique scheme. For example, for the estimation of the effective compliance $\mathbf{M}^*$ as a first step it is necessary to solve the elastic problem for a single inclusion in an infinite matrix (e.g. to find the tensor $\mathbf{B}(\mathbf{x})$, see e.g. Section 5) and in a second step, the single constant tensor $\mathbf{Y}$ is found from the purely elastic problem ($\boldsymbol{\beta}(\mathbf{x}) \equiv 0$) for the composite with ellipsoidal inclusions with the tensor $\mathbf{R}$ (3.14). Analogously the problem of evaluating the effective tensors $\boldsymbol{\beta}^*$, $U^{*\mathrm{II}}$ and the average stresses inside the components (4.4), (4.6) can be fully solved if one of the two parameters, $\mathbf{Y}$ or $\mathbf{M}^*$, is found (for example experimentally). For a proof, assume that $\mathbf{M}^*$ is known. Then (4.9) yields

$$\mathbf{Y} = \left(\mathbf{M}^* - \mathbf{M}^{(0)}\right)\left(\mathbf{R}n^{(1)}\right)^{-1} \tag{4.12}$$

and therefore,

$$\boldsymbol{\beta}^* = \boldsymbol{\beta}^{(0)} + \left(\mathbf{M}^* - \mathbf{M}^{(0)}\right)\mathbf{R}^{-1}\mathbf{F}, \tag{4.13}$$

$$U^{*\mathrm{II}} = -\frac{1}{2}\langle\boldsymbol{\beta}_1(\mathbf{x})\mathbf{B}(\mathbf{x})\rangle\mathbf{R}^{-1}\left[\mathbf{M}^* - \mathbf{M}^{(0)} - \mathbf{R}n^{(1)}\right]\left(\mathbf{R}n^{(1)}\right)^{-1}\mathbf{F} - \frac{1}{2}\langle\boldsymbol{\beta}_1(\mathbf{x})\mathbf{C}(\mathbf{x})\rangle. \tag{4.14}$$

The eqn (4.13) can be rewritten in an alternative form:

Theorem. In the framework of the hypothesis **H1**, the ratio of the increment of the effective thermal expansion to the increment of the effective stiffness is a constant tensor defined by the averaging solution for a single coated inclusion in an infinite matrix (3.14) and does not depend on the concrete statistically homogeneous microstructure of the while composite material and the detail microstructure of the individual inclusions as well:

$$\left(\mathbf{M}^* - \mathbf{M}^{(0)}\right)^{-1}\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}^{(0)}\right) = \mathbf{R}^{-1}\mathbf{F}. \tag{4.15}$$

In particular, for homogeneous (i.e. noncoated) inclusions the constant tensors **B**, **C**, **R** and **F** according to (3.15) and (3.16) are determined by the Eshelby tensor **S** and the jumps of the material property tensors $\mathbf{M}_1^{(1)}$, $\boldsymbol{\beta}_1^{(1)}$. Then from (4.13) and (4.14) the classical results for two-phase composites are derived

$$\boldsymbol{\beta}^* = \boldsymbol{\beta}^{(0)} + \left(\mathbf{M}^* - \mathbf{M}^{(0)}\right)\left(\mathbf{M}^{(1)} - \mathbf{M}^{(0)}\right)^{-1}\left(\boldsymbol{\beta}^{(1)} - \boldsymbol{\beta}^{(0)}\right), \tag{4.16}$$

$$U^{*\mathrm{II}} = \frac{1}{2}\left(\boldsymbol{\beta}^{(0)} - \boldsymbol{\beta}^{(1)}\right)\left(\mathbf{M}^{(1)} - \mathbf{M}^{(0)}\right)^{-1}\left(\boldsymbol{\beta}^* - \langle\boldsymbol{\beta}\rangle\right) \tag{4.17}$$

(see Levin, 1967; Rosen and Hashin, 1970; Kreher, 1988; Kreher and Pompe, 1989; where more references may be found). It is not surprising that the exact relations (4.16) and (4.17) are derived from the approximate ones, i.e. (4.13) and (4.14), since the additional assumption **H1** does not expand the class of the considered materials and homogenization methods. The representations (4.16) and (4.17) are formally invariant with respect to the replacement $v^{(1)} \leftrightarrow v^{(0)}$, although this cannot be said about the relations (4.13) and (4.14), obtained for matrix structure composites with ellipsoidal inclusions.

It should be mentioned that all results in Section 4 were obtained in the framework of hypothesis **H1** only. No restrictions are imposed on the concrete form of the tensor **Y** (4.1), on the microtopology of the coated inclusions or on the inhomogeneity of the stress state in the inclusions (4.5). Moreover, the assumption of an ellipsoidal shape of the inclusion in the hypothesis **H1**, was used only in order to obtain analytical solutions (3.10). This is because the tensor $\langle \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\rangle_i$ (3.12) is apparently homogeneous for $\mathbf{x}, \mathbf{y} \in v_i$ for an ellipsoid. For non-ellipsoidal inclusions one could assume that in some parts of the region $v_i^c \subset v_i$ the properties $\mathbf{M}_1(\mathbf{x}) \equiv 0$, $\boldsymbol{\beta}_1(\mathbf{x}) \equiv 0$, i.e. it is sufficient to include a real non-ellipsoidal inclusion $v_i \backslash v_i^c$ into an ellipsoid (with smallest possible volume) and call it the inclusion v_i with a 'coating' v_i^c. The further scheme for calculating the tensors **B**, **C** (3.10) and overall properties is the same, but the prescribed conditional distributions $\varphi(v_m|\mathbf{x}_m; \mathbf{x}_1, \ldots, \mathbf{x}_n)$will have a larger correlation hole $\cup v_{0i}$ $(i = 1, \ldots, n)$ than in the real composite material. This will result in an underestimation of the computed values of $\mathbf{M}^*$ for inclusions which are softer than the matrix and in an overestimation in the opposite case.

It should be mentioned that we do not pursue the goal to present in Appendix A all known methods based on hypothesis **H1**. There are, of course, several other methods which are based on hypothesis **H1** and no ranking between them is given here. Our main objective is to prove the general relations (4.13) and (4.14) which are valid in the framework of hypothesis **H1** only. No restrictions are imposed on the concrete statistically homogeneous microstructure of the whole composite material with a single sort of coated inclusions being analyzed as well as on the microtopology of coated inclusions or on the inhomogeneity of the stress state in the inclusions.

5. Single ellipsoidal inclusion with thin coating

5.1. General representation

In this section a possible application of the above general relations is presented. An analytical method for estimating the tensors $\mathbf{B}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$, see (3.13), is carried out for the example of a single ellipsoidal inclusion with a thin coating in an infinite matrix loaded by a constant macroscopic stress $\boldsymbol{\sigma}^0$.

Let the coated inclusion v_1 consist of an ellipsoidal core $v^i \subset v_1$ with a characteristic function $V^i(\mathbf{x})$ and thermoelastic parameters $\mathbf{M}^i$, $\boldsymbol{\beta}^i \equiv$ const and a thin coating $v^c \equiv v_1 \backslash v^i$ with a characteristic function $V^c \equiv V_1 - V^i$ and thermoelastic inhomogeneous properties $\mathbf{M}^c(\mathbf{x})$, $\boldsymbol{\beta}^c(\mathbf{x}) \neq$ const (see Fig. 1). In the considered case of a single inclusion the origin of the coordinate system is chosen to be the center of the inclusion $\mathbf{x}^i = \mathbf{0}$ and the coordinate axes coincide with the axes of the inclusions. In addition to (2.4) we define the jump of the material properties $\mathbf{f}$ $(\mathbf{f} = \mathbf{M}, \boldsymbol{\beta})$ across the boundary s^i between the core and the coating as $\mathbf{f}_2(\mathbf{x}) \equiv \mathbf{f}^i - \mathbf{f}^c(\mathbf{x})$.

For the single coated inclusion eqn (3.1) yields

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^0 + \int \Big[V^i(\mathbf{y}) + V^c(\mathbf{y})\Big] \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\big[\mathbf{M}_1(\mathbf{y})\boldsymbol{\sigma}(\mathbf{y}) + \boldsymbol{\beta}_1(\mathbf{y})\big]\,\mathrm{d}\mathbf{y}. \tag{5.1}$$

In analogy to Chang and Cheng (1992) as well as Cherkaoui et al. (1995) we find an approximative solution of eqn (5.1) under the approximative assumption of a homogeneous stress state in the core

$$\boldsymbol{\sigma}(\mathbf{x}) \equiv \boldsymbol{\sigma}^i = \text{ const} \quad \mathbf{x} \in v^i \subset v_1 \tag{5.2}$$

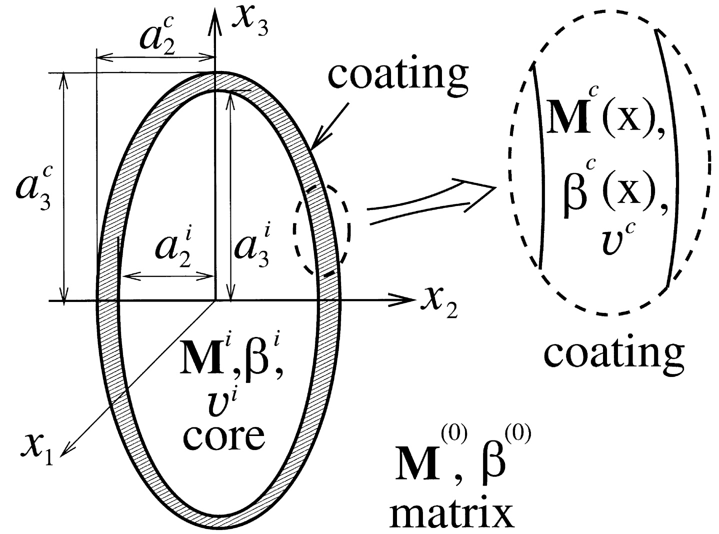

Fig. 1. Single coated inclusion.

and the thin-layer hypothesis, which means that the characteristic function $V^c(\mathbf{y})$ can be replaced by a surface δ-function (with weighting function ρ) at the outer surface s^i_- of the boundary $s^i = s^i_- \cup s^i_+$ and the volume integral of the continuous function $\mathbf{g}(\mathbf{y}), \mathbf{y} \in v^c$ is equal to a surface integral over outer surface s^i_- (see e.g. Gel'fand and Shilov, 1964):

$$\int V^c(\mathbf{y})\mathbf{g}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int S^i_-(\mathbf{s})\mathbf{g}(\mathbf{s})\rho\,\mathrm{d}\mathbf{s}, \tag{5.3}$$

where the product of the characteristic function S^i_- of the boundary s^i_- and some continuous function $\mathbf{g}(\mathbf{s}) \equiv \lim \mathbf{g}(\mathbf{y})$, $(\mathbf{y} \to \mathbf{s},\ \mathbf{y} \in v^c,\ \mathbf{s} \in s^i_-)$ is integrated over the surface s^i_-. In the particular case considered hereafter, the weighting function ρ for a domain v^c bounded by two ellipsoidal surfaces with the same center and with identically oriented semi-axes a_j and $a_j^c (j = 1, 2, 3)$, respectively, is estimated by Cherkaoui et al. (1995) by

$$\rho(\mathbf{y}) = \left(\frac{y_1^2}{a_1^4} + \frac{y_2^2}{a_2^4} + \frac{y_3^2}{a_3^4}\right)^{-1/2} \sum_{j=1}^{3} \frac{\left(a_j^c - a_j\right)}{a_j}\frac{y_j^2}{a_j^2}, \quad \left(\mathbf{y} \equiv (y_1, y_2, y_3)^{\mathrm{T}} \in s^i_-\right). \tag{5.4}$$

Under these assumptions the integral eqn (5.1) is, after averaging over the domain v^i, reduced to

$$\boldsymbol{\sigma}^i = \boldsymbol{\sigma}^0 - \mathbf{Q}^i\left(\mathbf{M}_1^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^i\right) + \int S^i_-(\mathbf{s})\mathbf{T}^i(\mathbf{x}^i - \mathbf{s})\left[\mathbf{M}_1^c(\mathbf{s})\boldsymbol{\sigma}(\mathbf{s}) + \boldsymbol{\beta}_1^c(\mathbf{s})\right]\rho\,\mathrm{d}\mathbf{s}. \tag{5.5}$$

Here and in the following the upper index i for the tensors $\mathbf{Q}^i$, $\mathbf{B}^i$, $\mathbf{T}^i(\mathbf{x}^i - \mathbf{s})$ stands for the calculation of these tensors for the core v^i by the use of the formulae (3.12), (3.15) and (3.9), respectively. Obviously, discarding the integral term in (5.5) leads to the Eshelby solution.

Taking the properties of the interface operator $\boldsymbol{\Gamma}(\mathbf{n})$ (4.7) (see Appendix B) into account leads to

$$\boldsymbol{\sigma}^c(\mathbf{s}) \equiv \boldsymbol{\sigma}(\mathbf{s}) = \boldsymbol{\sigma}^i + \boldsymbol{\Gamma}(\mathbf{n}, \mathbf{M}^c)\left[\mathbf{M}_2^i(\mathbf{s})\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i(\mathbf{s})\right], \tag{5.6}$$

$$\bar{v}^i \mathbf{T}^i\,(\mathbf{x}^i - \mathbf{s}) = \boldsymbol{\Gamma}\big(\mathbf{n}, \mathbf{M}^{(0)}\big) - \mathbf{Q}^i, \tag{5.7}$$

$$\boldsymbol{\Gamma}\big(\mathbf{n}, \mathbf{M}^{(0)}\big)\mathbf{M}_1^c\boldsymbol{\Gamma}(\mathbf{n}, \mathbf{M}^c) = \boldsymbol{\Gamma}\big(\mathbf{n}, \mathbf{M}^{(0)}\big) - \boldsymbol{\Gamma}(\mathbf{n}, \mathbf{M}^c), \tag{5.8}$$

where $\mathbf{n}$ is a unit outward normal vector on ∂v^i in the point $\mathbf{s}$. Here both interface operators $\boldsymbol{\Gamma}(\mathbf{n}, \mathbf{M}^{(0)})$ and $\boldsymbol{\Gamma}(\mathbf{n}, \mathbf{M}^c)$ are defined by formula (B.12) (see Appendix B) applied with the compliances $\mathbf{M}^{(0)}$ and $\mathbf{M}^c$, respectively; $\mathbf{M}_2^i(\mathbf{s}) \equiv \mathbf{M}^i - \mathbf{M}^c(\mathbf{s})$, $\boldsymbol{\beta}_2^i(\mathbf{s}) \equiv \boldsymbol{\beta}^i - \boldsymbol{\beta}^c(\mathbf{s})$.

Now eqn (5.5) reduces to an equation with only one unknown constant tensor $\boldsymbol{\sigma}^i$

$$\boldsymbol{\sigma}^i = \boldsymbol{\sigma}^0 + \left[\frac{\bar{v}^c}{\bar{v}^i}\,\mathbf{Q}^i - \mathbf{Q}_1\right]\big(\mathbf{M}_1^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^i\big) - \frac{1}{\bar{v}^i}\,\mathbf{Q}^i\int S_-^i(\mathbf{s})\big[\mathbf{M}_1^c(\mathbf{s})\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^c(\mathbf{s})\big]\rho\,\mathrm{d}\mathbf{s}$$

$$-\frac{1}{\bar{v}^i}\int S_-^i(\mathbf{s})\big[\mathbf{I} + \mathbf{Q}^i\mathbf{M}_1^c(\mathbf{s})\big]\Big[\mathbf{I} + \boldsymbol{\Gamma}\big(\mathbf{n}, \mathbf{M}^{(0)}\big)\mathbf{M}_1^c(\mathbf{s})\Big]^{-1}\boldsymbol{\Gamma}\big(\mathbf{n}, \mathbf{M}^{(0)}\big)\big[\mathbf{M}_2^i(\mathbf{s})\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i(\mathbf{s})\big]\rho\,\mathrm{d}\mathbf{s}. \tag{5.9}$$

The tensor $\mathbf{Q}_1$ for the ellipsoidal inclusion v_1 is determined by the relation (3.12).

This way we obtain an estimation of the stress distribution inside the coated inclusion $\boldsymbol{\sigma}^i$ and $\boldsymbol{\sigma}^c(\mathbf{s})$, see (5.6) and (5.9). Therefore, the stress concentration tensors $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$ in eqn (3.13) are found to be $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x}) =$ const at $\mathbf{x} \in v^i$ and $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x}) \neq$ const at $\mathbf{x} \in v^c$. After that the tensors $\mathbf{R}$ and $\mathbf{F}$ are defined by eqn (3.14) and the thermoelastic properties of the fictitious homogeneous inclusions $\mathbf{M}_1^{f(i)}$ and $\boldsymbol{\beta}_1^{f(i)}$, are evaluated by the relations either (3.17) or (3.18). Hence, the thermoelastic problem for the single coated inclusion is completely solved and we can come to the estimation of the overall thermoelastic properties, $\mathbf{M}^*$, $\boldsymbol{\beta}^*$, $U^{*\mathrm{II}}$, see (4.9)–(4.11) and average stresses $\langle\boldsymbol{\sigma}\rangle^{(0)}$ (4.4), $\langle\boldsymbol{\sigma}\rangle_1(\mathbf{x})$ (4.5), $\langle\boldsymbol{\sigma}^-(\mathbf{n})\rangle_x$ (4.8) inside the components by the use of different tensors $\mathbf{Y}$ (for some particular methods such tensors are represented in Appendix A).

Let us consider a simplification of the solution (5.9) for different particular cases of coated inclusions. According to (5.3) and (B.13) for a homogeneous coating, i.e. $\mathbf{M}^c(\mathbf{x})$ and $\boldsymbol{\beta}^c(\mathbf{x})$ are constant for any $\mathbf{x} \in v^c$, we get from (5.9) and (5.6)

$$\boldsymbol{\sigma}^i = \boldsymbol{\sigma}^0 - \mathbf{Q}_1\big[\mathbf{M}_1^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^i\big] - \frac{\bar{v}^c}{\bar{v}^i}\big\{\mathbf{Q}^i\mathbf{M}_1^c\mathbf{Q}^i(\mathbf{M}^c) + \mathbf{Q}^i(\mathbf{M}^c) - \mathbf{Q}^i\big\}\big[\mathbf{M}_2^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i\big]$$

$$+\big(\mathbf{I} + \mathbf{Q}^i\mathbf{M}_1^c\big)\big[\mathbf{Q}_1(\mathbf{M}^c) - \mathbf{Q}^i(\mathbf{M}^c)\big]\big[\mathbf{M}_2^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i\big], \tag{5.10}$$

$$\langle\boldsymbol{\sigma}^c\rangle^c = \boldsymbol{\sigma}^i + \left\{\mathbf{Q}^i(\mathbf{M}^c) + \frac{\bar{v}^c}{\bar{v}^i}\big[\mathbf{Q}_1(\mathbf{M}^c) - \mathbf{Q}^i(\mathbf{M}^c)\big]\right\}\big[\mathbf{M}_2^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i\big]. \tag{5.11}$$

Here the tensors $\mathbf{Q}$ and $\mathbf{Q}(\mathbf{M}^c)$ (with the indices i and 1) are calculated for the compliances $\mathbf{M}^{(0)}$ and $\mathbf{M}^c$, respectively; $\langle(\bullet)\rangle^c$ denotes average over v^c. In the particular case of homothetic surfaces ∂v^i and ∂v^c (when $\mathbf{Q}^i = \mathbf{Q}_1$) the eqns (5.10) and (5.11) can be further simplified to become

$$\boldsymbol{\sigma}^i = \boldsymbol{\sigma}^0 - \mathbf{Q}^i\big[\mathbf{M}_1^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^i\big] - \frac{\bar{v}^c}{\bar{v}^i}\big[\mathbf{Q}^i\mathbf{M}_1^c\mathbf{Q}^i(\mathbf{M}^c) + \mathbf{Q}^i(\mathbf{M}^c) - \mathbf{Q}^i\big]\big[\mathbf{M}_2^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i\big], \tag{5.12}$$

$$\langle\boldsymbol{\sigma}^c\rangle^c = \boldsymbol{\sigma}^i + \mathbf{Q}^i(\mathbf{M}^c)\big[\mathbf{M}_2^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_2^i\big]. \tag{5.13}$$

The relations (5.10)–(5.13) have been proposed previously by a more specific method by Cherkaoui et al. (1995) (see also Buryachenko and Rammerstorfer, 1996, 1999).

Clearly, the thin-layer hypothesis can be rejected if the elastic properties of the coating and the matrix are the same: $\mathbf{M}^c(\mathbf{x}) \equiv \mathbf{M}^{(0)}$, $\mathbf{x} \in v^c$. Then under assumption (5.2) the eqn (5.1) can be solved immediately leading to

$$\boldsymbol{\sigma}^i = \mathbf{B}^i\left[\boldsymbol{\sigma}^0 - \mathbf{Q}^i\boldsymbol{\beta}_1^i \quad + \int V^c(\mathbf{y})\mathbf{T}^i\left(\mathbf{x}^i - \mathbf{y}\right)\boldsymbol{\beta}_1^c(\mathbf{y})\,\mathrm{d}\mathbf{y}\right], \quad \mathbf{B}^i = \left(\mathbf{I} + \mathbf{Q}^i\mathbf{M}_1^i\right)^{-1}, \tag{5.14}$$

$$\langle\boldsymbol{\sigma}^c\rangle^c = \boldsymbol{\sigma}^0 + \frac{\bar{v}^i}{\bar{v}^c}\left(\mathbf{Q}^i - \mathbf{Q}_1\right)\left[\mathbf{M}_1^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^i\right] - \mathbf{Q}_1\langle\boldsymbol{\beta}_1^c\rangle^c - \frac{\bar{v}^i}{\bar{v}^c}\int V^c(\mathbf{y})\mathbf{T}^i\left(\mathbf{x}^i - \mathbf{y}\right)\boldsymbol{\beta}_1^c(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{5.15}$$

For a thin coating we obtain from (5.9) and (5.6)

$$\boldsymbol{\sigma}^i = \mathbf{B}^i\left[\boldsymbol{\sigma}^0 - \mathbf{Q}^i\boldsymbol{\beta}_1^i - \frac{\bar{v}^c}{\bar{v}^i}\mathbf{Q}^i\langle\boldsymbol{\beta}_1^c\rangle^c + \frac{1}{\bar{v}^i}\int S_-^i(\mathbf{s})\boldsymbol{\Gamma}\left(\mathbf{n}, \mathbf{M}^{(0)}\right)\boldsymbol{\beta}_1^c(\mathbf{s})\rho\,\mathrm{d}\mathbf{s}\right], \tag{5.16}$$

$$\langle\boldsymbol{\sigma}^c\rangle^c = \boldsymbol{\sigma}^0 + \frac{\bar{v}^i}{\bar{v}^c}\left(\mathbf{Q}^i - \mathbf{Q}_1\right)\left[\mathbf{M}_1^i\boldsymbol{\sigma}^i + \boldsymbol{\beta}_1^i\right] - \frac{\bar{v}^c}{\bar{v}^i}\mathbf{Q}^i\langle\boldsymbol{\beta}_1^c\rangle^c + \left(\frac{1}{\bar{v}^i} - \frac{1}{\bar{v}^c}\right)\int S_-^i(\mathbf{s})\boldsymbol{\Gamma}\left(\mathbf{n}, \mathbf{M}^{(0)}\right)\boldsymbol{\beta}_1^c(\mathbf{s})\rho\,\mathrm{d}\mathbf{s}. \tag{5.17}$$

These relations result from (5.14) and (5.15) under the assumption of the thin-layer hypothesis (5.3).

5.2. Numerical assessment of thin-layer hypothesis

Let us consider a single spherical inclusion of the radius a^i with a homothetic spherical coating of the radius a^c with $\mathbf{Q}^i = \mathbf{Q}_1$ and according to (5.4), $\rho = a^c - a^i$ in an infinite matrix. The elastic properties of the coating coincide with the elastic properties of the isotropic matrix, i.e.

$$\mathbf{L}^c = \mathbf{L}^{(0)} = \left(3k^{(0)}, 2\mu^{(0)}\right) \equiv 3k^{(0)}\mathbf{N}_1 + 2\mu^{(0)}\mathbf{N}_2, \quad \mathbf{N}_1 \equiv \boldsymbol{\delta}\otimes\boldsymbol{\delta}/3, \quad \mathbf{N}_2 \equiv \mathbf{I} - \mathbf{N}_1. \tag{5.18}$$

Let $\boldsymbol{\beta}^i = \boldsymbol{\beta}^{(0)} = 0$ and $\boldsymbol{\beta}^c$ has a special form with some physical meaning represented by

$$\boldsymbol{\beta}^c = \gamma\boldsymbol{\delta} + (\chi - \gamma)\mathbf{n}\otimes\mathbf{n}, \tag{5.19}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂v^i.

It can easily be shown that γ and χ are the transformation parameters of the coating in the tangential and normal directions, respectively. If $\gamma = \chi$ this constitutive characterization of the coating corresponds to the particular case of an isotropical thermal expansion, considered by a number of authors (see e.g. Hatta and Taya, 1987; Chang and Cheng, 1992). We will analyze a less trivial case $\gamma \neq \chi$ allowing the existence of a prestress in the coating. This situation is typical for the case of production of coated inclusions separately from the matrix. Moreover, under purely thermal deformations (e.g. $\boldsymbol{\sigma}^0 = \mathbf{0}$) the plastic strains of the matrix near the inclusions also have the form (5.19) (see e.g. Buryachenko et al., 1997). Clearly, in some local coordinate system connected with the inclusion surface ∂v^i the tensor $\boldsymbol{\beta}^c$ is constant. However, in the global coordinate system $\boldsymbol{\beta}^c$ is a function of the unit normal $\mathbf{n}$ and therefore, is an inhomogeneous function of the coordinates. Therefore, the system (5.10) and (5.11), obtained under the assumption of a homogeneous coating, is not suitable and it is necessary to consider the system for either a thin coating, i.e. (5.16) and (5.17), or a thick one, i.e. (5.14) and (5.15). In the more general case of a thick coating we have

$$\boldsymbol{\sigma}^i = \mathbf{B}^i\left[\boldsymbol{\sigma}^0 + (\chi-\gamma)\int V^c(\mathbf{y})\mathbf{T}^i\big(\mathbf{x}^i-\mathbf{y}\big)(\mathbf{n}\otimes\mathbf{n})\,\mathrm{d}\mathbf{y}\right], \tag{5.20}$$

$$\langle\boldsymbol{\sigma}^c\rangle^c = \boldsymbol{\sigma}^0 - \frac{1}{3}(\chi+2\gamma)\mathbf{Q}^i\boldsymbol{\delta} - \frac{\bar v^i}{\bar v^c}(\chi-\gamma)\int V^c(\mathbf{y})\mathbf{T}^i\big(\mathbf{x}^i-\mathbf{y}\big)(\mathbf{n}\otimes\mathbf{n})\,\mathrm{d}\mathbf{y}. \tag{5.21}$$

According to (3.14) we can now find the concentration tensors

$$\mathbf{B} = \mathbf{I} + \frac{\bar v^i}{\bar v_1}(\mathbf{B}^i - \mathbf{I}), \tag{5.22}$$

$$\mathbf{C} = -\frac{1}{3}\frac{\bar v^c}{\bar v_1}(\chi+2\gamma)\mathbf{Q}^i\boldsymbol{\delta} + \frac{\bar v^i}{\bar v_1}(\chi-\gamma)(\mathbf{B}^i-\mathbf{I})\int V^c(\mathbf{y})\mathbf{T}^i\big(\mathbf{x}^i-\mathbf{y}\big)(\mathbf{n}\otimes\mathbf{n})\,\mathrm{d}\mathbf{y}. \tag{5.23}$$

Clearly, the tensor $\mathbf{C}$ (5.23) is an isotropic one: $\mathbf{C}\equiv C\boldsymbol{\delta}$, as well as $\mathbf{C}(\mathbf{x})\equiv C^i\boldsymbol{\delta}$, $\mathbf{x}\in v^i$ and

$$\mathbf{C}(\mathbf{x}) \equiv C^i\boldsymbol{\delta} = (\chi-\gamma)\mathbf{B}^i\int V^c(\mathbf{y})\mathbf{T}^i\big(\mathbf{x}^i-\mathbf{y}\big)(\mathbf{n}\otimes\mathbf{n})\,\mathrm{d}\mathbf{y}, \quad \text{at } \mathbf{x}\in v^i. \tag{5.24}$$

In order to show the good quality of the estimation resulting from the thin-layer hypothesis let us define a normalized residual average stress ($\sigma^0\equiv 0$) in the coated inclusion $\sigma^{\mathrm{res}}\equiv -\langle\sigma_{11}\rangle^{(1)}/[\gamma 3\mathbf{Q}^k(1-\mathbf{B}^{ik})]$, with isotropic tensors $\mathbf{Q}^i=\mathbf{Q}_1=(3Q^k,\ 2Q^\mu)$, $\mathbf{B}^i=(3B^{ik},\ 2B^{i\mu})$. Average residual stress in the coating inclusion $\langle\boldsymbol{\sigma}\rangle^{(1)}$ can be found by the use of eqn (4.3) with $\mathbf{Y}=\mathbf{I}$ (when $\langle\boldsymbol{\sigma}\rangle^{(1)}\equiv\mathbf{C}$). We consider now the particular case of rigid spherical inclusions inside the coating with pure tension prestress ($\chi=0$) and $v^{(0)}=0.3$. Using the thin-layer hypothesis (5.3) from (5.23) the normalized residual stress is determined to become $\sigma^{\mathrm{res}}=\bar v^c/\bar v_1$ which does not depend on the elastic properties of the matrix. In Fig. 2 the parameter σ^{res} is represented as a function of relative thickness of the coating $h=(a^c-a^i)/a^i$. Results obtained by using the thin-layer hypothesis (5.3) are compared with results (5.23) without this approximative assumption. As becomes evident from Fig. 2 the thin-layer hypothesis provides an

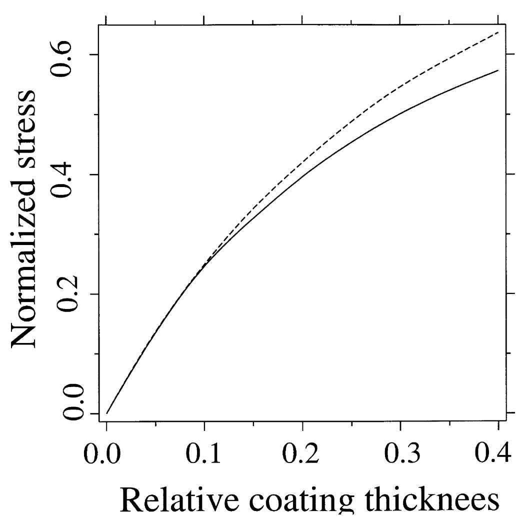

Fig. 2. Variation of the normalized average residual stress in the coated inclusion, $\boldsymbol{\sigma}^{\mathrm{res}}$, as a function of the relative coating thickness $h=(a^c-a^i)/a^i$. Dotted line: under assumption of thin-layer hypothesis; solid line: without this assumption.

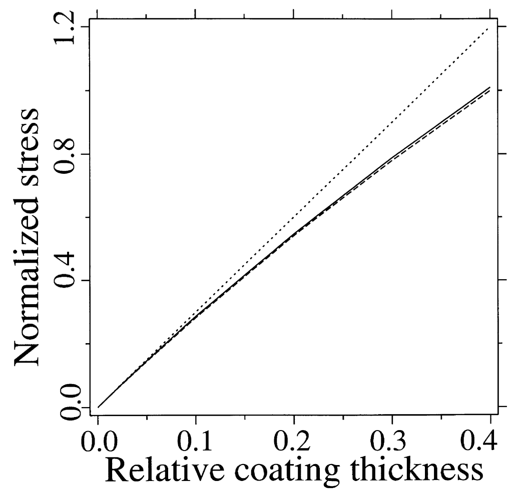

Fig. 3. Normalized residual stress in the core, $\sigma_i^{\rm res}$, calculated analytically by the use of the thin-layer hypothesis (5.7) (with $v^c/v^i = 3h$, dotted line) and without this approximative assumption (dashed line). Numerical evaluation by the finite element method (solid line).

acceptable exactness for not too thick coating, let us say for $h < 0.2$. For increasing Poisson ratios of the matrix $\nu^{(0)}$, the solid line approaches the dashed line and for $\nu^{(0)} = 0.5$ both curves coincide.

Now the normalized residual stress in the core $\sigma_i^{\rm res} \equiv \langle\sigma_{11}\rangle^i/(3B^{ik}Q^k\gamma)$ is estimated, where the average over the core v^i is considered: $\langle(^\bullet)\rangle^i$. Analytically derived results (5.24) under thin-layer hypothesis (5.3) (for which $v^c/v^i \cong 3h$) are compared in Fig. 3 with the one obtained by more exact approach eqn (5.24) and with results from the finite element analyses. The numerical results obtained from finite element analyses (presented in the paper by Buryachenko and Rammerstorfer, 1996) differs from the analytical solution (5.24) by not more than 1%. Increase of the Poisson ratio of the matrix, $\nu^{(0)}$, would move the solid and dashed lines in Fig. 3 slightly but not significantly; clearly, the thin-layer approximation $\sigma_i^{\rm res} = 3h$ does not change. In conclusion, the analytical solution (5.23) and (5.24) provides high exactness in the considered examples.

6. Discussion

Let us discuss the main hypotheses as well as the limitations of the proposed estimations and their possible generalizations.

The assumption of homogeneity of $\bar{\boldsymbol{\eta}}(\mathbf{x})$, $(\mathbf{x} \in v_i)$ is a classical hypothesis of micromechanics (see the earliest references by Lax, 1951) and was required in order to make it easier to solve the algebraic system (3.20) and (A.4) (see Appendix A), which, in principle, can also be solved for a polynomial function $\bar{\boldsymbol{\sigma}}_i(\mathbf{x})$, $\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\ldots,}$ n in analogy to Moschovidis and Mura (1975) or Chen and Acrivos (1978). Then in the case of the rejection of the hypothesis **H1** it is necessary to introduce new concentration tensors of larger dimension in addition to **B**, **C**, **F**, **R** (3.10) and (3.11), which significantly complicates the calculations and reduces the generality of the obtained formulae (4.9)–(4.11).

However, it should be noted that for rigid spherical inclusions in an incompressible matrix the solution by Chen and Acrivos (1978) $\mu^*/\mu^{(0)} = 1 + 2.5c + 5.01c^2$ is very close to that obtained with the dilute approximation by MEFM (Buryachenko and Lipanov, 1986) $\mu^*/\mu^{(0)} = 1 + 2.5c + 4.85c^2$ (see also Buryachenko, 1996). Furthermore, Buryachenko and Parton (1992b) as well as Buryachenko (1999a) estimated the exactness of the MEFM by comparing their solution $\mathbf{L}^*(c)$ with the exact analytical solution by Sangani and Lu (1987) for simple cubic packing of either rigid or vacuole spherical inclusions. For simple cubic packing the maximum inclusion concentration equals 0.52. However, even for $c = 0.50$ the error of the MEFM results does not exceed 15%. Therefore, the exactness of the hypothesis **H1** should be considered as sufficient for practical purposes.

The principal limitation of this paper is due to the assumption of statistical homogeneity of the composite microstructure, although Buryachenko and Parton (1990a), Buryachenko (1998, 1999b) and Buryachenko and Rammerstorfer (1998a) analyzed the case of statistically inhomogeneous composites, i.e. when the inclusion concentration is a function of the coordinates i.e. $\varphi(v_i) = \varphi(v_i)(\mathbf{x})$ (such materials have been denoted as some sorts of graded materials). In this case the general equation (3.19) is valid if the average obtained from (2.6) is understood as the ensemble average in the considered point $\mathbf{x}$. Even under the hypothesis **H1** and some additional simplifications Buryachenko (1998), as well as Buryachenko and Rammerstorfer (1998a) obtained the dependence of $\mathbf{M}^*$ on the inclusion concentration $\varphi(v_i)(\mathbf{x})$ and (what is less trivial) the nonlocal character of such a dependence. Then even for constant external loading $\boldsymbol{\sigma}^0$ the effective field $\langle\boldsymbol{\eta}\rangle(\mathbf{x})$ is a function of the coordinates and the ensemble average stress (4.3)–(4.5) will be represented in the form of either integral or differential operators (Buryachenko, 1998; Buryachenko and Rammerstorfer, 1998a; see also Buryachenko, 1999b).

The possible constitutive relations are not limited to the case of thermo-elasticity (2.3). It is only important that the response of an inclusion is defined by the eqns (3.10), notwithstanding the inclusion can be considered as some sort of a ‘black box’. Hence, the response of materials with gas-saturated pores or cracks has the same form (3.8) and (3.9) and we can obtain estimations for the overall properties immediately (as opposed to Buryachenko and Lipanov, 1986) from the solution of the purely elastic problem. Further the constitutive relation of thermo-electro-magneto-elasticity has the form (2.3), however with larger dimension. Such analogy is valid for both the general equation (3.20) and for a single inclusion (3.8) and (3.9) and was used by Buryachenko and Parton (1992b), Buryachenko and Shermergor (1995) for obtaining overall property estimations and some exact relations for composites with homogeneous inclusions generalizing the elastic results by Rosen and Hashin (1970), Kreher and Pompe (1989), Buryachenko and Kreher (1995). In the light of the results obtained in this paper, obviously, consideration of coated inclusions with thermo-electromagneto-elastic components are possible too.

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Appendix A. The evaluation of the Y (4.1)

Effective medium method

The additional closing hypothesis of the effective medium method is described as follows: each inclusion in the composite behaves as an isolated one in a homogeneous medium whose properties coincide with the effective properties. Formally this means

$$\mathbf{H2}:\quad \mathbf{Y} \equiv \mathbf{I},\quad \mathbf{M}^* = \mathbf{M}^{(0)} + \mathbf{R}(\mathbf{M}^*)n^{(1)}, \tag{A.1}$$

where the tensor $\mathbf{R}(\mathbf{M}^*)$ is calculated by the use of the formulae (3.11) with $\mathbf{M}^{(0)}$ being replaced by $\mathbf{M}^*$; this exchange applies to the formulae (4.3)–(4.11), too.

The so-called differential scheme of constructing effective elastic moduli also belongs to the class of the effective medium methods (see e.g. Norris et al., 1985). This scheme is considered as a process of consecutive additions of infinitesimal values of the inclusion phase in a uniform medium with a modulus equal to the effective modulus of the medium with the previous additions of inclusions to the matrix, which yields the closed-form equation

$$\mathbf{H2}:\quad \mathbf{Y} \equiv \mathbf{I},\quad \frac{\mathrm{d}\mathbf{M}^*}{\mathrm{d}c^{(1)}} = \frac{1}{(1 - c^{(1)})\bar{v}_1}\mathbf{R}(\mathbf{M}^*). \tag{A.2}$$

Mori–Tanaka method

According to the closing hypothesis of the Mori–Tanaka method each inclusion in the composite is considered as an isolated one, located inside an infinite matrix and loaded by the effective field $\langle\bar{\boldsymbol{\sigma}}\rangle_i \equiv \langle\boldsymbol{\sigma}\rangle^0$. Then from the equation $c^{(1)}\mathbf{B}\langle\boldsymbol{\sigma}\rangle^{(1)} + c^{(0)}\langle\boldsymbol{\sigma}\rangle^{(0)} = \boldsymbol{\sigma}^0$ we obtain

$$\mathbf{H2}:\quad \mathbf{Y}^{-1} = \mathbf{I} + (\mathbf{B} - \mathbf{I})c^{(1)}. \tag{A.3}$$

Multiparticle effective field method

By using hypothesis **H1**, the system (3.20) with fixed values $\langle\bar{\boldsymbol{\sigma}}(\mathbf{x})_{1,\dots,}\ n\rangle$ for $\mathbf{x} \in v_i$ on the right-hand-side of the equations becomes algebraic when the solution (3.10) for one inclusion in the field $\langle\bar{\boldsymbol{\sigma}}(\mathbf{x})\rangle_i$ $(i = 1, \dots, n)$ is applied:

$$\langle\bar{\boldsymbol{\sigma}}(\mathbf{x})\rangle_i = \langle\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\dots,\,n}\rangle_i + \sum_{j=1}^{n}\big(1 - \delta_{ij}\big)\mathbf{T}_{ij}\big(\mathbf{R}\langle\bar{\boldsymbol{\sigma}}(\mathbf{y})\rangle_j + \mathbf{F}\big), \tag{A.4}$$

where δ_{ij} is a Kronecker symbol. Let us define a matrix $\mathbf{Z}^{-1}$ with elements $(\mathbf{Z}^{-1})_{ij}(i,\ j = 1, \dots, n)$ (the elements represent the fourth-order tensors)

$$(\mathbf{Z}^{-1})_{ij} = \mathbf{I}\boldsymbol{\delta}_{ij} - \big(1 - \boldsymbol{\delta}_{ij}\big)\mathbf{R}\mathbf{T}_{ij}(\mathbf{x}_i - \mathbf{x}_j). \tag{A.5}$$

Then the system (A.4) can be rewritten as follows

$$\mathbf{R}\langle\bar{\boldsymbol{\sigma}}\rangle_i + \mathbf{F} = \sum_{j=1}^{n}\mathbf{Z}_{ij}\big[\mathbf{R}\langle\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\dots,\,n}\rangle_j + \mathbf{F}\big]. \tag{A.6}$$

The system (3.20) can be solved by analytical methods, if the hypothesis

$$\mathbf{H2}:\quad \langle\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,\,2}\rangle_i = \langle\bar{\boldsymbol{\sigma}}(\mathbf{x})\rangle_i = \text{const}\quad (i = 1, 2) \tag{A.7}$$

is assumed. This independence of $\langle\tilde{\boldsymbol{\sigma}}(\mathbf{x})_{1,2}\rangle$ on the spacing between the inclusions v_1 and v_2 occurs for the large distance between the inclusions. Then from (3.20), taking (3.10), (A.6) into account, we get

$$\langle\bar{\boldsymbol{\sigma}}\rangle_i = \int \mathbf{T}_{iq}(\mathbf{x}_i - \mathbf{x}_q)\mathbf{Z}_{qi}\varphi\big(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i\big)\, \mathrm{d}\mathbf{x}_q(\mathbf{R}\langle\bar{\boldsymbol{\sigma}}_i + \mathbf{F})$$

$$+ \int\Big[\mathbf{T}_{iq}(\mathbf{x}_i - \mathbf{x}_q)\mathbf{Z}_{qq}\varphi\big(v_q, \mathbf{x}_q|; v_i; \mathbf{x}_i\big) - \mathbf{T}_i(\mathbf{x}_i - \mathbf{x}_q)n^{(1)}\Big] \cdot \big(\mathbf{R}\langle\bar{\boldsymbol{\sigma}}\rangle_q + \mathbf{F}\big)\, \mathrm{d}\mathbf{x}_q, \tag{A.8}$$

where the matrix elements $\mathbf{Z}_{qi}$, $\mathbf{Z}_{qq}$ are nondiagonal elements and diagonal ones of the binary interaction matrix $\mathbf{Z}$ (A.5) for the two inclusions v_q and v_i. This set of equations can be solved for $\langle\bar{\boldsymbol{\sigma}}\rangle_i$:

$$\mathbf{R}\langle\bar{\boldsymbol{\sigma}}\rangle_i + \mathbf{F} = \mathbf{Y}(\mathbf{R}\langle\boldsymbol{\sigma}\rangle + \mathbf{F}), \tag{A.9}$$

where the matrix $\mathbf{Y}$ determines the action of the surrounding inclusions on the considered one and has an inverse matrix $\mathbf{Y}^{-1}$ given by

$$\mathbf{Y}^{-1} = \mathbf{I} \quad - \mathbf{R}\int \mathbf{T}_{iq}(\mathbf{x}_i - \mathbf{x}_q)\mathbf{Z}_{qi}\varphi\big(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i\big)\, \mathrm{d}\mathbf{x}_q$$

$$-\mathbf{R}\int\Big[\mathbf{T}_{iq}(\mathbf{x}_i - \mathbf{x}_q)\mathbf{Z}_{qq}\varphi\big(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i\big) - \mathbf{T}_i(\mathbf{x}_i - \mathbf{x}_q)n^{(1)}\Big]\, \mathrm{d}\mathbf{x}_q. \tag{A.10}$$

Buryachenko and Parton (1992a) proposed a differential version of the MEFM, in which at each step of the differential scheme (A.2) a problem of n interacting inclusions inside some effective medium is solved.

The ‘quasi-crystalline’ approximation by Lax (1951) (see also Kunin, 1983) expressed as

$$\textbf{H2}: \quad \mathbf{Z}_{ij} = \mathbf{I}\delta_{ij}, \tag{A.11}$$

leads to the one particle approximation of MEFM by Buryachenko and Parton (1992)

$$\mathbf{Y}^{-1} = \mathbf{I} - \mathbf{R}\int\Big[\mathbf{T}_{iq}(\mathbf{x}_i - \mathbf{x}_q)\varphi\big(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i\big) - \mathbf{T}_i(\mathbf{x}_i - \mathbf{x}_q)n^{(1)}\Big]\, \mathrm{d}\mathbf{x}_q. \tag{A.12}$$

Under a point approximation of the inclusions (exact for infinitely spaced heterogeneities) we have

$$\mathbf{T}_{iq}(\mathbf{x}_i - \mathbf{x}_q) = \mathbf{T}_i(\mathbf{x}_i - \mathbf{x}_q) = \boldsymbol{\Gamma}(\mathbf{x}_i - \mathbf{x}_q), \tag{A.13}$$

and from (A.12) one receives

$$\mathbf{Y}^{-1} = \mathbf{I} - \mathbf{R}\int \boldsymbol{\Gamma}(\mathbf{x}_i - \mathbf{x}_q)\Big[\varphi\big(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i\big) - n^{(1)}\Big]\, \mathrm{d}\mathbf{x}_q. \tag{A.14}$$

The representation (A.14) follows from the results obtained by Levin (1976) and by the use of the variational method by Willis (1977), Ponte Castañeda and Willis (1995) (see also Buryachenko and Rammerstorfer, 1998b), who considered in detail the case of multicomponent composites and the effect of the spatial distribution of the homogeneous inclusions. Only in particular cases, in which the shape of the correlation hole v_{0i} is homothetic to the inclusion shape v_i the formulae (A.3) and (A.14) coincide.

Appendix B. Properties of the interface operator $\boldsymbol{\Gamma}$ (n) (4.7)

According to Hill (1983) we define the projective operators $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\mathbf{E}$, $\mathbf{F}$ of the second- and fourth-order, respectively, as follows:

$$\tau_{kl} \equiv n_k n_l, \quad v_{kl} \equiv \delta_{kl} - \tau_{kl},$$

$$F_{klmn} \equiv (v_{km} v_{ln} + v_{lm} v_{kn})/2, \quad E_{klmn} \equiv I_{klmn} - F_{klmn}. \tag{B.1}$$

Furthermore, the surface tensors are defined by

$$\mathbf{L}(\mathbf{n})^{\pm} = \mathbf{L}^{\pm}\boldsymbol{\tau}, \quad \mathbf{G}(\mathbf{n})^{\pm} = \left[\mathbf{L}(\mathbf{n})^{\pm}\right]^{-1}, \quad \boldsymbol{\Gamma}(\mathbf{n})^{\pm} = \mathbf{L}^{\pm} - \mathbf{L}^{\pm}\mathbf{U}(\mathbf{n})^{\pm}\mathbf{L}^{\pm},$$

$$U(n)^{\pm}_{klmn} = \left[n_k G(n)^{\pm}_{lm} n_n\right]_{(kl)(mn)}. \tag{B.2}$$

Here and below the symbols + and − relate to the different boundary sides.

By testing we immediately obtain 'orthogonal' properties of the operators defined in (B.1)

$$\boldsymbol{\tau\tau} = \boldsymbol{\tau}, \quad \boldsymbol{vv} = \boldsymbol{v}, \quad \boldsymbol{v\tau} = \mathbf{0},$$

$$\mathbf{FF} = \mathbf{F}, \quad \mathbf{EE} = \mathbf{E}, \quad \mathbf{E}\boldsymbol{v} = 0, \quad \mathbf{F}\boldsymbol{\tau} = \mathbf{0}, \quad \mathbf{FE} = \mathbf{0}. \tag{B.3}$$

Hence the tensors $\mathbf{U}(\mathbf{n})$, $\boldsymbol{\Gamma}(\mathbf{n})$ in (B.2) can be expressed in terms of the projective operators (B.1)

$$\mathbf{U}(\mathbf{n}) = [\mathbf{ELE}]^{-1}\mathbf{E}, \quad \boldsymbol{\Gamma}(\mathbf{n}) = [\mathbf{FMF}]^{-1}\mathbf{F}. \tag{B.4}$$

Perfect contact between two materials means

$$\mathbf{E}\boldsymbol{\sigma}^{+} = \mathbf{E}\boldsymbol{\sigma}^{-}, \tag{B.5}$$

$$\mathbf{F}\boldsymbol{\varepsilon}^{+} = \mathbf{F}\boldsymbol{\varepsilon}^{-}. \tag{B.6}$$

The following relations between the stress tensors near the interface are involved (see e.g. Buryachenko and Kreher, 1995):

$$\boldsymbol{\sigma}^{-} = \boldsymbol{\sigma}^{+} + \boldsymbol{\Gamma}(\mathbf{n})^{-}\left[(\mathbf{M}^{+} - \mathbf{M}^{-})\boldsymbol{\sigma}^{+} + \left(\boldsymbol{\beta}^{+} - \boldsymbol{\beta}^{-}\right)\right] \tag{B.7}$$

$$\boldsymbol{\sigma}^{+} = \boldsymbol{\sigma}^{-} + \boldsymbol{\Gamma}(\mathbf{n})^{+}\left[(\mathbf{M}^{-} - \mathbf{M}^{+})\boldsymbol{\sigma}^{-} + \left(\boldsymbol{\beta}^{-} - \boldsymbol{\beta}^{+}\right)\right]. \tag{B.8}$$

Substitution of (B.7) into the right-hand-side of (B.8) leads to

$$\boldsymbol{\Gamma}(\mathbf{n})^{-} - \boldsymbol{\Gamma}(\mathbf{n})^{+} = \boldsymbol{\Gamma}(\mathbf{n})^{-}(\mathbf{M}^{+} - \mathbf{M}^{-})\boldsymbol{\Gamma}(\mathbf{n})^{+}. \tag{B.9}$$

Let an ellipsoidal inclusion v^i with the homogeneous compliance $\mathbf{M}^{+}$ be located in an infinite homogeneous matrix with compliance $\mathbf{M}^{-}$ and loaded by the homogeneous stress $\boldsymbol{\sigma}^0$. Then, according to Eshelby's theorem (with $\boldsymbol{\beta} \equiv 0$), we have

$$\boldsymbol{\sigma}^{+} = \boldsymbol{\sigma}^0 - \mathbf{Q}^i(\mathbf{M}^{+} - \mathbf{M}^{-})\boldsymbol{\sigma}^{+}, \tag{B.10}$$

$$\boldsymbol{\sigma}^{-} = \boldsymbol{\sigma}^0 + \bar{v}^i\mathbf{T}^i(\mathbf{x}_i - \mathbf{x}^{-})(\mathbf{M}^{+} - \mathbf{M}^{-})\boldsymbol{\sigma}^{+}, \tag{B.11}$$

where the tensor $\mathbf{Q}^i$ of the inclusion v^i is associated with the Eshelby tensor $\mathbf{S}^i$ (3.12) by $\mathbf{S}^i = \mathbf{I} - \mathbf{M}^{-}\mathbf{Q}^i$ and the tensor $\mathbf{T}^i(\mathbf{x}^i - \mathbf{x}^{-})$ is defined by the relation (3.9) for the point $\mathbf{x}^{-} \notin v^i$ near the ellipsoidal surface ∂v^i. Substituting the relations (B.7) and (B.11) into eqn (B.10) we obtain

$$\bar{v}^i \mathbf{T}_i(\mathbf{x}_i - \mathbf{x}^-) = \boldsymbol{\Gamma}(\mathbf{n})^- - \mathbf{Q}^i. \tag{B.12}$$

Let us consider a coated inclusion $v_1 = v^i \cup v^c$ with a characteristic function $V_1 = V^i + V^c$. According to (3.9) the tensor $\boldsymbol{\Gamma}(\mathbf{n})^-$ in (B.12) is integrated over the coating v^c

$$\int V^c(\mathbf{y}) \boldsymbol{\Gamma}(\mathbf{n})^- \mathrm{d}\mathbf{y} = \bar{v}^c \mathbf{Q}^i + \int \Big[V_1(\mathbf{y}) - V^i(\mathbf{y}) \Big] \int V^i(\mathbf{x}) \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y}. \tag{B.13}$$

Changing the integration sequence and applying Eshelby's theorem, we get from (B.13)

$$\int V^c(\mathbf{y}) \boldsymbol{\Gamma}(\mathbf{n})^- \mathrm{d}\mathbf{y} = \bar{v}^c \mathbf{Q}^i + \bar{v}^i \big(\mathbf{Q}^i - \mathbf{Q}_1 \big). \tag{B.14}$$

In particular for an isotropic medium with the elastic moduli

$$\mathbf{L} = (3k, 2\mu) \equiv 3k\mathbf{N}_1 + 2\mu\mathbf{N}_2, \quad \mathbf{N}_1 \equiv \boldsymbol{\delta} \otimes \boldsymbol{\delta}/3, \quad \mathbf{N}_2 \equiv \mathbf{I} - \mathbf{N}_1, \tag{B.15}$$

the inversion of the matrix $\mathbf{L}(\mathbf{n})$ may be simplified and we obtain

$$L(n)_{kl} = \mu \delta_{kl} + \left(k + \frac{\mu}{3} \right) n_k n_l, \quad G(n)_{kl} = \mu^{-1} \left(\delta_{kl} - \frac{2k + \mu}{3k + 4\mu} n_k n_l \right),$$

$$U(n)_{klmn} = \frac{1}{2\mu} \left(E_{klmn} - \frac{3k - 2\mu}{3k + 4\mu} n_k n_l n_m n_n \right),$$

$$\boldsymbol{\Gamma}(n)_{klmn} = 2\mu \left[F_{klmn} + \frac{3k - 2\mu}{3k - +4\mu} (\delta_{kl} - n_k n_l)(\delta_{mn} - n_m n_n) \right]. \tag{B.16}$$

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